national accelerator laboratory

TM-313-A
0402

ERRATUM AND ADDENDUM TO TM-313

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Both Eqs. (1) and (9) are approximate equations. For Eq. (1) the approximation assumes that $\frac{x}{\beta} << 1$ and terms of the second and higher orders in $\frac{x}{\beta}$ are neglected. These conditions are satisfied for the example cases given on pp. 3 and 4.

For Eq. (9) the approximation assumes that $\frac{\Delta\beta}{\beta} << 1$ and $\frac{\Delta B'}{B'} << 1$, and terms of the second and higher orders in $\frac{\Delta\beta}{\beta}$ and $\frac{\Delta B'}{B'}$ are neglected. These conditions are <u>not</u> satisfied for the example cases given on pp. 6 and 7. The results are, therefore, <u>invalid</u>.

Eq. (10) shows that $\frac{\Delta\beta}{\beta}$ (if << 1) is a sinusoidal function of θ with amplitude $\sqrt{U}$. For $\sqrt{U} > 1$, then, at some θ-locations $\frac{\Delta\beta}{\beta} < -1$ and the modified $\overline{\beta} = \beta + \Delta\beta < 0$ which is certainly not meaningful. This is another indication that Eq. (9) and its solution Eq. (10) are invalid when $\frac{\Delta\beta}{\beta} = \sqrt{U} > 1$.

For the case of one δ-function focusing bump the exact solution can be obtained using the transfer matrix. The transfer matrix around the entire closed orbit plus the bump (ε_o) is

$$\begin{pmatrix} 1 & 0 \\ -\varepsilon_o & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos 2\pi\nu + \begin{pmatrix} \alpha_o & \beta_o \\ -\gamma_o & -\alpha_o \end{pmatrix} \sin 2\pi\nu \right]$$

$$= \begin{pmatrix} 1 & 0 \\ -\varepsilon_o & 1 \end{pmatrix} \cos 2\pi\nu + \begin{pmatrix} \alpha_o & \beta_o \\ -(\gamma_o+\varepsilon_o\alpha_o) & -(\alpha_o+\varepsilon_o\beta_o) \end{pmatrix} \sin 2\pi\nu$$

where, as before, $\varepsilon_o \equiv \frac{(\Delta B'\ell)_o}{B\rho}$. The modified "tune" $\bar{\nu}$ and β-function at the bump $\bar{\beta}_o$ are, therefore, given by

$$\begin{cases} \cos 2\pi\bar{\nu} = \cos 2\pi\nu - \dfrac{\varepsilon_o\beta_o}{2} \sin 2\pi\nu \\ \bar{\beta}_o \sin 2\pi\bar{\nu} = \beta_o \sin 2\pi\nu . \end{cases} \tag{1A}$$

As ε_o varies from zero to either positive or negative values stability limits $\cos 2\pi\bar{\nu} = \pm 1$ will be encountered at certain values of ε_o. Beyond these values of ε_o, $|\cos 2\pi\bar{\nu}| > 1$ and the motion is unstable. At the stability limits the modified β-function $\bar{\beta}$ is ∞ everywhere except at discrete θ-locations where $\bar{\beta} = 0$, namely $\frac{\Delta\beta}{\beta} \equiv \frac{\bar{\beta}-\beta}{\beta}$ is ∞ everywhere except at these discrete θ-locations where $\frac{\Delta\beta}{\beta} = -1$. Although at the stability limit this exact $\frac{\Delta\beta}{\beta}$ is hardly sinusoidal, one may expect that the stability limits correspond roughly to $\sqrt{U} = 1$ when the "approximate" $\bar{\beta}$ as given by Eq. (10) also goes to zero at these discrete θ-locations. Eq. (13) gives, then, for the stability limits

$$\frac{\varepsilon_o\beta_o}{2} = \pm \sin 2\pi\nu \quad \text{"approximate"} \tag{2A}$$

while the exact conditions are given by Eq. (1A) as

$$\frac{\varepsilon_o\beta_o}{2} = \frac{\cos 2\pi\nu \mp 1}{\sin 2\pi\nu}$$

$$= - \frac{1}{\cos 2\pi\nu \pm 1} \sin 2\pi\nu. \quad \text{exact} \qquad (3A)$$

The exact and the "approximate" conditions are identical when ν = (integer) $\pm \frac{1}{4}$.

For the main ring $\nu \cong 20\frac{1}{4}$. Both Eqs. (2A) and (3A) give for the stability limits

$$\frac{\varepsilon_o\beta_o}{2} = \pm 1$$

or, for $\beta_o \cong 100$ m

$$\varepsilon_o = \pm\frac{2}{\beta_o} \cong \pm 0.02 \text{ m}^{-1}.$$

Missing one quadrupole ($\varepsilon_o = \pm 0.04$ m^{-1}) will take us <u>beyond the stability limit</u>. The most we can tolerate is missing $\frac{1}{2}$ of a quadrupole.

The "invariant" U is clearly also an approximate invariant valid only when $\frac{\Delta\beta}{\beta} << 1$. We can put U in a more conventional form.

$$U = \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4\nu^2}\left[\frac{d}{d\theta}\left(\frac{\Delta\beta}{\beta}\right)\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \frac{1}{4}\left[\beta\frac{d}{dz}\left(\frac{\Delta\beta}{\beta}\right)\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \left[-\frac{\beta'}{2}\frac{\Delta\beta}{\beta} + \frac{(\Delta\beta)'}{2}\right]^2$$

$$= \left(\frac{\Delta\beta}{\beta}\right)^2 + \alpha^2 \left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)^2 \tag{4A}$$

where prime means $\frac{d}{dz}$ and $\alpha = -\frac{\beta'}{2}$, $\Delta\alpha = -\frac{(\Delta\beta)'}{2}$.

D. A. Edwards gave the exact form of this invariant as

$$U = \frac{\left(\frac{\Delta\beta}{\beta}\right)^2 + \alpha^2 \left(\frac{\Delta\beta}{\beta} - \frac{\Delta\alpha}{\alpha}\right)^2}{1 + \frac{\Delta\beta}{\beta}}. \tag{5A}$$

His derivation is given below: Consider two locations 1 and 2 around the closed orbit with no focusing bump in between. The transfer matrices from locations 1 and 2 all the way around the closed orbit are respectively

$$\begin{aligned}\overline{M}_1 &= \cos 2\pi\overline{\nu} + \overline{J}_1 \sin 2\pi\overline{\nu} \\ &= \cos 2\pi\overline{\nu} + (J_1+\Delta J_1) \sin 2\pi\overline{\nu}\end{aligned}$$

and

$$\begin{aligned}\overline{M}_2 &= \cos 2\pi\overline{\nu} + \overline{J}_2 \sin 2\pi\overline{\nu} \\ &= \cos 2\pi\overline{\nu} + (J_2+\Delta J_2) \sin 2\pi\overline{\nu}.\end{aligned}$$

Writing the transfer matrix from location 1 to location 2 as M_{12} (there is no need for a bar on top because there is no bump between locations 1 and 2) the relation $\overline{M}_2 = M_{12}M_1M_{12}^{-1}$ leads to

$$J_2+\Delta J_2 = M_{12}(J_1+\Delta J_1)M_{12}^{-1}.$$

Remembering that $J_2 = M_{12}J_1M_{12}^{-1}$ we get

$$\Delta J_2 = M_{12}\Delta J_1 M_{12}^{-1}$$

which shows that the determinant of ΔJ is invariant within a bump-free region. We can, thus, write

$$U = -|\Delta J| = (\Delta\alpha)^2 - (\Delta\beta)(\Delta\gamma) = \text{invariant.}$$

Substituting

$$\Delta\gamma = \frac{1+(\alpha+\Delta\alpha)^2}{\beta+\Delta\beta} - \frac{1+\alpha^2}{\beta}$$

$$= -\frac{\frac{\Delta\beta}{\beta} + \alpha^2 \left[\frac{\Delta\beta}{\beta} - 2\frac{\Delta\alpha}{\alpha} - \left(\frac{\Delta\alpha}{\alpha}\right)^2\right]}{\beta\left(1+\frac{\Delta\beta}{\beta}\right)}$$

we get directly the expression (5A).

I am grateful to Dr. S. Ohnuma for pointing out the error in TM-313 and to Dr. D. Edwards for the derivation of the exact expression of the invariant U, and to both of them for several illuminating discussions.